RELATIVE ROBUSTNESS
OF SEVERAL CEP ESTIMATORS

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THESIS

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OF SEVERAL CEP ESTIMATORS

by

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This thesis presents a discussion of the problems involved in estimation of Circular Error Probable (CEP). Several estimators are compared through simulation, under two models; the power model and the Rayleigh model. Several measures of effectiveness are calculated for each of the competing estimators. It is found that maximum likelihood estimation based on the power distribution performs well for "heavy tailed"

(20. ABSTRACT Continued)

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I. INTRODUCTION

Evaluation of a weapons system is frequently an expensive and time consuming task. Only relatively few tests may be allowed because of the high cost of testing. It is possible that a good weapon system could be rejected because of inefficient utilization of the small amount of data available. Hence, efficient use of the limited data is required to reduce this risk.

Several authors have considered the measurement of weapon delivery accuracy using Circular Error Probable (CEP). This measure of accuracy is defined to be the radius of a circle centered at the target which on the average contains fifty percent of impact points, that is, the median radial miss distance. A functional relationship between CEP and the distribution parameters is given by the following integral:

$$F(\mathrm{CEP}) = \int_0^{\mathrm{CEP}} f(r)\,dr = 0.5 ,$$

where f(r) is the probability density of radial miss distance.

Various parametric models for the distribution of impact points about the target have been used in connection with such estimations. A discussion of these models is given by Jordan [1]. But, deficiencies in the models have been

noted. Specifically, they tend to provide poor fit to the "upper tail" of the radial distributions. That is, they provide less than satisfactory explanation of impacts observed far from the target. A commonly used model is the "Rayleigh model", in which impacts in the target plane are assumed to be distributed as the unbiased circular normal distribution. One might anticipate that Rayleigh-based CEP estimators might not perform adequately with a large proportion of actual test data. Several potential competitors to Rayleigh CEP estimators were therefore developed for comparison.

Barr and Jordan [2] developed and proposed the power distribution model which is based on an infinite mixture of Rayleigh radial miss distance distributions. This distribution has properties compatible with the interpretation of σ in the underlying Rayleigh distribution model; they suggested the use of maximum likelihood estimates of the parameters of the power distribution which in turn provide a maximum likelihood estimate (MLE) of CEP.

The jackknife is a technique which was proposed by Quenouille [10] for reducing bias in estimation problems. We tried the jackknife procedure to estimate CEP under the Rayleigh model.

The sample median was also used to estimate CEP. Since CEP is defined as the median radial miss distance, a natural estimator for CEP is the sample median.

To measure the performance of the estimators, a variety of data sets from known distributions, such as uniform, Rayleigh, power and Weibull were generated by simulation. Samples from the known distributions were generated to test the capability of the estimators to handle widely disparate situations.

II. MAXIMUM LIKELIHOOD ESTIMATION BASED ON THE POWER DISTRIBUTION

Barr and Jordan [1], [2] have given a discussion of CEP accuracy measurement under a power distribution model. They showed that estimating the parameters of the power distribution by the method of moments may not be adequate; moreover it is possible only when the ratio of the squared sample mean and variance $\bar{r}^2/s_r^2$ does not exceed $\pi/4 - \pi$. Consequently, they suggested use of the maximum likelihood (M.L.) method.

Analytically closed expressions for the M.L. estimators cannot be obtained, the solution of normal equations yielding the parameter estimates requires numerical techniques. In this section we examine a numerical method to obtain the M.L. estimates of the parameters of the power distribution, which in turn provide a M.L. estimate of CEP. These estimation procedures require use of a computer.

A. MATHEMATICAL FORMULATION

The probability density function of the radial miss distance R having a power distribution with parameters κ and ζ is

$$f_R(r;\kappa,\zeta) = \frac{2\kappa r}{\zeta} \left(\frac{\zeta}{\zeta + r^2}\right)^{\kappa+1} \quad ; \quad r \geq 0,\ \kappa,\zeta > 0$$

A C.D.F. for radial miss distance is

$$F_R(r;\kappa,\zeta) = 1 - \left(\frac{\zeta}{\zeta + r^2}\right)^{\kappa} \quad ; \quad r \geq 0, \quad \kappa,\zeta > 0 \qquad (1)$$

The CEP is given by

$$CEP = F_R^{-1}(\tfrac{1}{2}) = [(2^{\frac{1}{\kappa}} - 1)\zeta]^{\frac{1}{2}}$$

Because radial miss distance is always non-negative, one can equivalently consider squared radial miss distance. In the case of the power model, this leads to some notational simplification. Let Y denote the squared miss distance R^2. Then from (1),

$$F_Y(y) = 1 - (1 + \tfrac{y}{\zeta})^{-\kappa} \qquad ; \quad y \geq 0 \ , \ \kappa,\zeta > 0$$

$$f_Y(y) = \tfrac{\kappa}{\zeta}(1 + \tfrac{y}{\zeta})^{-\kappa-1} \qquad ; \quad y \geq 0 \ , \ \kappa,\zeta > 0 \qquad (2)$$

Here ζ and κ are parameters to be estimated from observed impact data. The relationship between CEP and these parameters is

$$CEP^2 = F_Y^{-1}(\tfrac{1}{2}) = [(2^{\frac{1}{\kappa}} - 1)\zeta]$$

The method of maximum likelihood is based upon the likelihood function $\phi(y)$ which is defined as the joint density function of a sample of size n; i.e.,

$$\phi(y) = \prod_{i=1}^{n} f_Y(y_i) = \frac{\kappa^n}{\zeta^n} \prod_{i=1}^{n} \left(1 + \frac{y_i}{\zeta}\right)^{-\kappa-1}$$

Since it is easier to deal with sums rather than products when maximizing, $\phi(y)$ is transformed into

$$L(y) = \log \phi(y)$$

$$= n \log \kappa + n\kappa \log \zeta - (\kappa+1) \sum_{i=1}^{n} \log(\zeta+y_i)$$

There is no loss of generality here because the maximum of a positive function occurs at the same point as the maximum of the logarithm of the function. The normal equations are then

$$m(\kappa,\zeta) = \frac{\partial L(y)}{\partial \kappa} = \frac{n}{\kappa} + n\log \zeta - \sum_{i=1}^{n} \log(\zeta+y_i) = 0$$

and

$$g(\kappa,\zeta) = \frac{\partial L(y)}{\partial \zeta} = \frac{n\kappa}{\zeta} - (\kappa+1) \sum_{i=1}^{n} \frac{1}{\zeta + y_i} = 0$$

The solution of $m(\hat{\kappa},\hat{\zeta}) = 0$ and $g(\hat{\kappa},\hat{\zeta}) = 0$ gives the parameter M.L. estimators $\hat{\kappa}$, $\hat{\zeta}$. As mentioned before, in the present case, this requires numerical techniques.

B. NUMERICAL SOLUTIONS: NEWTON'S METHOD

This method is a Newton's iteration technique and the concept and procedure are discussed in [3]. Newton's method provides a statistical criterion for stopping the iterations.

The pure form of Newton's method is given by the following iteration scheme:

$$\hat{X}_{j+1} = \hat{X}_j - [F(\hat{X}_j)]^{-1} \nabla f(\hat{X}_j)^T$$

where

$$X_j = [\hat{\kappa}_j, \hat{\zeta}_j], \qquad f(X_j) = [\frac{\partial L(y)}{\partial \hat{\kappa}}, \frac{\partial L(y)}{\partial \hat{\zeta}}] \ .$$

First we compute

$$\frac{\partial L(y)}{\partial \hat{\kappa}} = \frac{n}{\hat{\kappa}} + n \log \hat{\zeta} - \sum_{i=1}^{n} \log (\hat{\zeta}+y_i)$$

$$\frac{\partial L(y)}{\partial \hat{\zeta}} = \frac{n\hat{\kappa}}{\hat{\zeta}} - (\hat{\kappa}+1) \sum_{i=1}^{n} \frac{1}{\hat{\zeta} + y_i}$$

$$\frac{\partial^2 L(y)}{\partial \hat{\kappa}^2} = \frac{-n}{\hat{\kappa}^2}$$

$$\frac{\partial^2 L(y)}{\partial \hat{\kappa} \partial \hat{\zeta}} = \frac{n}{\hat{\zeta}} - \sum_{i=1}^{n} \frac{1}{\hat{\zeta} + y_i}$$

$$\frac{\partial^2 L(y)}{\partial \hat{\zeta}^2} = \frac{-n\hat{\kappa}}{\hat{\zeta}^2} + (\hat{\kappa}+1) \sum_{i=1}^{n} \frac{1}{(\hat{\zeta} + y_i)^2} \ .$$

Let

$$D = \frac{\partial^2 L(y)}{\partial \hat{\kappa}^2} \cdot \frac{\partial^2 L(y)}{\partial \hat{\zeta}^2} - \left(\frac{\partial^2 L(y)}{\partial \hat{\kappa} \partial \hat{\zeta}}\right)^2$$

If D were identically equal to zero, this method would not be applicable. Now take

$$F^{-1}(\hat{X}) = \frac{\begin{matrix} \dfrac{\partial^2 L(y)}{\partial \hat{\zeta}^2} & -\dfrac{\partial^2 L(y)}{\partial \hat{\kappa} \partial \hat{\zeta}} \\ -\dfrac{\partial^2 L(y)}{\partial \hat{\kappa} \partial \hat{\zeta}} & \dfrac{\partial^2 L(y)}{\partial \hat{\kappa}^2} \end{matrix}}{D} .$$

In order to develop the termination criterion, consider the statistic for (for reference see [4])

$$B = \nabla f(\hat{X}_j) F^{-1}(\hat{X}_j) \nabla f(\hat{X}_j)^{-1}$$

B is asymptotically a Chi-square r. v. with 2 degrees of freedom, therefore it might be reasonable to continue the iteration until the statistic B becomes less than the 100α percentile point of the chi-square distribution with two degrees of freedom, $\chi^2(\alpha)$.

III. MAXIMUM LIKELIHOOD ESTIMATION BASED ON THE RAYELIGH DISTRIBUTION MODEL

Suppose impacts, measured in a suitable plane, have range (X) and deflection (Y) components of impact that are jointly uncorrected bivariate normal distributed with mean at the target and common unknown variance σ^2 in both directions. That is, for the present development, it will be assumed that the joint density of miss distances about the target is given by

$$f(x,y) = (2\pi\sigma^2)^{-1} e^{-(x^2+y^2)/2\sigma^2} \,, \quad -\infty < x,y < \infty .$$

It is easy to derive the density of radial miss distance. Let $R = (X+Y)^{1/2}$ and transform to polar coordinates to obtain

$$F_R(r) = P[R \leq r] = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^r R\, e^{-\frac{R^2}{2\sigma^2}} \, dR \, d\theta$$

$$= 1 - e^{-\frac{r^2}{2\sigma^2}} \; ; \quad r \geq 0, \quad \sigma^2 > 0 \; .$$

This distribution of the radial miss distance is referred to as either the radial normal distribution or the Rayleigh distribution.

Then the density function of the radial miss distance, $f_R(r)$ is

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad ; \quad r \geq 0, \quad \sigma^2 > 0 \, .$$

To find the relationship between CEP and σ, we calculate

$$\text{CEP} = F_R^{-1}(\tfrac{1}{2}) = \sqrt{2 \ln 2}\, \sigma \simeq 1.1774\, \sigma \, ,$$

which is a well known relation. Hence, if we want to estimate CEP with this model, first we might estimate the population standard deviation. The maximum likelihood estimator for σ is easily found to be

$$\hat{\sigma} = \left(\frac{1}{2n} \sum_{i=1}^{n} r_i^{\,2}\right)^{\frac{1}{2}} \, ,$$

where $r_1, \ldots, r_n$ is the sample of observed radial miss distances. This estimator has a slight bias, however. An adjusted unbiased estimator for CEP is

$$\hat{\text{CEP}} = 1.1774 \left(\sqrt{n}\; \frac{\Gamma(n)}{\Gamma(\frac{1}{2}(2n+1))}\right) \hat{\sigma}, \tag{3}$$

where the quantity in parenthesis is an unbiasing factor for $\hat{\sigma}$. It has been shown by Chapman and Robbins [6] that CEP = 1.1848 $\hat{\sigma}$ for sample size 20 is a best estimator of CEP under the normality and other assumptions stated above. However other estimators, while not as efficient as the best estimator, may have more robustness under variations in the assumed distribution of radial miss distance.

The term "jackknife" has been used by Miller [9] and Quenouille [10] to describe a method of modifying a biased estimator to bring it closer to an unbiased condition. From (3) we see the M.L. CEP estimator has a bias term of order $\frac{1}{\sqrt{n}}$. We are thus led to try to examine the jackknife method to reduce this bias in the estimator. Brillinger[7] has provided a discussion of the asymptotic properties of estimators obtained with the jackknife method when it is applied to M.L.E's.

The jackknife procedure is based on dividing the data into groups, obtaining estimates from combinations of the groups, and then averaging these estimates. We apply this to our problem, as follows. The maximum likelihood estimator for $\hat{\sigma}^2$ is

$$\hat{\sigma}^2 = \frac{1}{2n} \sum_{j=1}^{n} r_j^2$$

Divide these data into m groups, where $r_1, \ldots, r_n$ is the sample of n observed radial miss distances and the size of each subgroup is k. First estimate the parameter $\hat{\sigma}^2$ based on all data

$$\hat{\sigma}_o^2 = \frac{1}{2n} \sum_{j=1}^{n} r_j^2 \quad ;$$

next estimate $\hat{\sigma}^2$ based on all data except those in the i^{th} group

$$\hat{\sigma}_{-i}^{2} = \frac{1}{2n} \sum_{j \neq i}^{n} r_j^{2} \quad ; \quad i = 1, 2, \ldots, m.$$

Then compute the pseudo values

$$\hat{\sigma}_{i}^{2} = m\hat{\sigma}_{o}^{2} - (m-1)\hat{\sigma}_{-i}^{2} \quad ; \quad i = 1, 2, \ldots, m.$$

Finally, compute the average of the pseudo values,

$$\bar{\hat{\sigma}}^{2} = \frac{\sum_{i=1}^{m} \hat{\sigma}_{i}^{2}}{m}$$

and estimate the CEP by

$$\widehat{CEP} = \sqrt{2 \ln 2 \, \bar{\hat{\sigma}}^{2}}$$

IV. THE SAMPLE MEDIAN

Since CEP is defined as the median radial miss distance, a natural estimator for CEP is the sample median (m). By definition; the median is the 50^{th} percentile of the distribution, i.e.,

$$P(R \leq m) \geq 0.50 \quad \text{and} \quad P(R \geq m) \geq .50$$

Since F(CEP) = 0.50 and R is assumed to have a continuous distribution,

$$\hat{CEP} = m = F^{-1}(\tfrac{1}{2})$$

Suppose that the n radial miss distance observations of the sample are arranged in order of magnitude, $r_{(1)}, r_{(2)}, \ldots, r_{(n)}$. We shall consider only the estimates of the population median defined as follows:

(a) If the sample size is odd, and n = 2p+1, the median (m) is taken as the $(p+1)^{th}$ value.

(b) If the sample size is even, and n = 2p, the median (m) is taken as the midpoint between the p^{th} and $(p+1)^{th}$ values or

$$\tfrac{1}{2}(r_{(p)} + r_{(p+1)}) .$$

The distribution of the median will tend to be symmetrical, and should be very nearly normal in almost all populations. The distribution of the median in the case of small samples from a normal population, whether the sample size is odd or even, will tend to be normal [Hoji [11]). The normal form is approached very rapidly. The value of the standard deviations of the median for n even and n odd approach the same limit,

$$\sigma_m = \sqrt{\frac{\pi}{2}} \cdot \tilde{\sigma}/ n = 1.25331\ \tilde{\sigma}/\sqrt{n}$$

where $\tilde{\sigma}$ is the standard deviation in the population sampled.

We use the order statistics to examine the sample median under the Rayleigh distribution model (see Inselmann [12]), and find its bias and variance so as to enable one to make a comparison with other estimates. If n is even, the distribution of sample median is given by the following:

$$g(r_{(0.5n)}) = \{n!/(n-[0.5n])!([0.5n]-1)!\}f(r)F(r_{[0.5n]})^{[0.5n]-1}$$

$$\cdot\ [1 - F(r_{[0.5n]})]^{n-[0.5n]} \quad .$$

For convenience we let $a = 0.5n$, and

$$C = n!/(n-[0.5n])!([0.5n]-1)! = (2a)!/a!(a-1)!$$

so the sample median is r_a. Then we may write the distribution of sample median under the Rayleigh distribution model,

$$g(ra) = c[\frac{r}{\sigma^2} \exp(-\frac{r^2}{2\sigma^2})][1-\exp(-\frac{r^2}{2\sigma^2})]^{a-1}[\exp(-\frac{r^2}{2\sigma^2})]^{a}$$

$$= c \frac{r}{\sigma^2} [\exp(- \frac{r^2(a+1)}{2\sigma^2})][1 - \exp(-\frac{r^2}{2\sigma^2})]^{a-1}$$

We will now consider computation of the mean of ra,

$$E(ra) = c \int_0^{\infty} \frac{r^2}{\sigma^2}[\exp(-\frac{r^2(a+1)}{2\sigma^2})][1-\exp(- \frac{r^2}{2\sigma^2})]^{a-1} dr$$

$$= c \int_0^{\infty} \frac{r^2}{\sigma^2}[\exp(- \frac{r^2(a+1)}{2\sigma^2})] \sum_{K=0}^{a-1} (-1)^K \binom{a-1}{K} [\exp(-\frac{r^2}{2\sigma^2})]^K dr$$

$$= c \sum_{K=0}^{a-1} (-1)^K \binom{a-1}{K} \int_0^{\infty} \frac{r^2}{\sigma^2} \exp(- \frac{r^2(a+1+K)}{2\sigma^2}) dr$$

and make the following transformation

$$y = r^2(a+1+K)/2\sigma^2 ,$$

$$r = \sigma \sqrt{2y/(a+1+K)} ,$$

$$dy = [r(a+1+K)/\sigma^2] dr .$$

To obtain,

$$E(ra) = c\sigma\sqrt{2} \sum_{K=0}^{a-1} (-1)^K \binom{a-1}{K} / (a+1+K)^{\frac{3}{2}} \int_0^\infty \sqrt{y}\, e^{-y}\, dy$$

$$= c\sigma\Gamma(\tfrac{3}{2}) \cdot \sqrt{2} \cdot \sum_{K=0}^{a-1} \frac{(-1)^K \binom{a-1}{K}}{(a+1+K)^{3/2}} \ .$$

We can adjust the bias term and using the same transformation as above we may now find the second moment

$$E(r_a^{\,2}) = c \int_0^\infty \frac{r^3}{\sigma^2} \exp\left(-\frac{r^2(a+1)}{2\sigma^2}\right) \sum_{K=0}^{a-1} (-1)^K \binom{a-1}{K} \left[\exp\left(-\frac{r^2}{2\sigma^2}\right)\right]^K dr$$

$$= 2c\sigma^2 \sum_{K=0}^{a-1} \frac{(-1)^K \binom{a-1}{K}}{(a+1+K)^2} \ .$$

The variances of this sample median can now be computed using the well-known formula

$$\mathrm{Var}\ (r_a) = E(r_a^{\,2}) - [E(r_a)]^2 \ .$$

V. SIMULATION METHODOLOGY

The previous sections have been concerned with a development of different estimators for the median of the distribution of radial miss distances obtained under a set of fixed impact conditions. An analytical comparison is difficult to perform for these estimators, so computer simulation was used. This section summarizes the different models and estimators discussed in the previous sections, and includes an analysis of the results obtained from simulations. Although the sample problems do not represent actual weapon test results, an attempt has been made to generate data from a wide variety of "realistic" distributions. Therefore this simulation analysis should show relationships between our estimators of CEP, and indicate their relative robustness, and hence potential for use in a variety of applications. The alternative estimators are as follows:

$\widehat{CEP}$ 1: maximum likelihood unbiased Rayleigh estimator

$\widehat{CEP}$ 2: maximum likelihood Jackknife Rayleigh estimator, with subgroup sample size k = 1

$\widehat{CEP}$ 3: maximum likelihood power estimator under Newton's method

$\widehat{CEP}$ 4: sample median estimator

For the purpose of our simulation comparison of these four CEP estimators, we generated samples of radial miss distances (or, in some cases, squared radial miss distance)

from several parent distributions: Rayleigh, uniform, power and Weibull distributions. Throughout, we assumed a sample size of 20 impact observations. The method of generation was by C.D.F. inversion of generated uniform (0,1) values. For each parent distribution (hereafter called a "case") we replicated estimation of CEP's 500 times. Summaries of the sample distributions of 500 values of 5 CEP's were developed.

Eight cases were considered; in each, parameter values were used which gave a true CEP of 100. The cases are as follows (U is a uniform (0,1) random variable throughout):

Case I. (Rayleigh distribution)

$$F_{R^2}(y) = 1 - e^{-y/2\sigma^2} \quad \text{with} \quad \sigma^2 = 10^4/2\ln 2$$

$$CEP = \sqrt{2 \ln 2}\, \sigma$$

$$\text{Generated value:} \quad Y = -2\sigma^2 \ln U = -14426.95 \ln U$$

Case II. (Uniform distribution)

$$F_R(y) = \frac{1}{200} y \quad , \quad \text{with} \quad y \in (0,200).$$

$$CEP = 100$$

$$\text{Generated value } Y = 200\, U \,.$$

Case III. (Power distribution)

$$F_{R^2}(y) = 1 - (1 + \frac{y}{\zeta})^{-\kappa} \; ; \quad y \geq 0 \; , \quad \zeta,\kappa > 0$$

with $\kappa = 1$ and $\zeta = 10^4$

$$\mathrm{CEP} = [\zeta(2^{\frac{1}{\kappa}} - 1)]^{\frac{1}{2}}$$

$$\text{Generated value } Y = (U^{-\frac{1}{\kappa}} - 1)\zeta = 10^4(\frac{1}{U} - 1)$$

Case IV. (Translated Rayleigh distribution)

$$F_{R^2}(y) = 1 - e^{-(Y-100)/2\sigma^2}$$

where $2\sigma^2 = 14282.6809$

$$\mathrm{CEP} = [100 + 2\sigma^2 \ln 2]^{\frac{1}{2}}$$

$$\text{Generated value } Y = -2\sigma^2 \ln U + 100$$

Cases V-VIII. (Weibull distributions)

$$F_{R^2}(y) = 1 - e^{-y^{\frac{w}{v}}}$$

$$\mathrm{CEP} = (v \ln 2)^{\frac{1}{2w}}$$

Generated value Y in the table below

Four sets of parameters (w,v) were selected to give CEP = 100 and to provide distributions with varying upper tail thickness. One case (with w = 1) gives an exponential distribution with mean v, and this corresponds to the Rayleigh case. Case I above may be considered also as a Weibull case. The parameter pairs used, in order of increasing upper tail size, are given in the following table.

Case	w	v	$Y = (-v\ln U)^{\frac{1}{w}}$
V	2	$10^8/\ln 2$	$Y = (-144260504.1 \ln U)^{\frac{1}{2}}$
VI	1.5	$10^6/\ln 2$	$Y = (-1442695.04 \ln U)^{\frac{2}{3}}$
I	1.0	$10^4/\ln 2$	$Y = -4426.95 \ln U$
VII	.5	$10^2/\ln 2$	$Y = (-144.27 \ln U)^2$
VIII	.1	$10^{.4}/\ln 2$	$Y = (-3.623886 \ln U)^{10}$

VI. SIMULATION RESULTS

As mentioned above, we tested four CEP estimators from eight different cases. Major comparison measures include bias, variance and mean square error. Others are included in the tables that follow.

CASE I. RAYLEIGH DISTRIBUTION

$$Y = -\ 14426.95 \ln U$$

MOE	$\hat{CEP}$ 1*	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
MEAN	99.93	99.94	95.96	107
STD DEV	11.66	11.68	12.22	38.8
MSE	136.03	136.37	165.77	1549
GED MEAN	99.25	99.25	95.17	100
VARIANCE	136.03	137.37	149.45	1500
COEFF VAR	0.12	0.12	0.13	0.36
SKEWNESS	0.15	0.14	0.049	0.35
KURTOSIS	0.10	0.085	0.307	0.34
RANGE	71.38	70.87	83.60	252
MINIMUM	67.25	67.29	53.74	17.7
.10 QUANTILE	84.82	84.80	80.99	58.4
.25	92.12	92.09	87.67	82.5
.50	99.63	99.63	95.96	105
.75	107.49	107.34	104.00	132
.90	114.83	114.83	111.32	156
MAXIMUM	138.63	138.16	137.34	269

* $\hat{CEP}$ 1: maximum likelihood unbiased Rayleigh estimator.

$\hat{CEP}$ 2: maximum likelihood Jackknife Rayleigh estimator, with subgroup sample size k = 1.

$\hat{CEP}$ 3: maximum likelihood power estimator under Newton's Second method.

$\hat{CEP}$ 4: sample median estimator.

CASE II. UNIFORM DISTRIBUTION

$$Y = (200\ U)^2$$

MOE	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
MEAN	98.61	96.52	91.24	102
STD DEV	9.87	9.73	17.08	41.7
MSE	109.32	107.03	368.63	1744.0
GED MEAN	96.09	96.01	*	91.5
VARIANCE	97.83	94.92	291.89	1740
COEFF VAR	0.10	0.10	0.19	0.40
SKEWNESS	- 0.21	- 0.20	- 1.83	0.005
KURTOSIS	0.61	0.60	5.54	- 0.56
RANGE	67.28	66.35	125.60	191
MINIMUM	59.52	59.97	0.0	3.75
.10 QUANTILE	84.19	84.25	72.1	45.7
.25	90.10	90.13	85.30	742
.50	96.84	96.74	94.26	102
.75	102.96	102.75	101.52	132
.90	108.66	108.40	107.62	158
MAXIMUM	126.80	126.32	125.60	194

* some $\hat{CEP}$ = 0.0

CASE III. POWER DISTRIBUTION

$$Y = 10^4 \cdot \left(\frac{1}{U} - 1\right)$$

MOE	$\widehat{CEP}$ 1	$\widehat{CEP}$ 2	$\widehat{CEP}$ 3	$\widehat{CEP}$ 4
MEAN	209.61	229.33	100.71	113
STD DEV	190.15	257.90	21.07	46.0
MSE	2460	83000	444.36	2279
GED MEAN	177.76	185.62	98.60	103
VARIANCE	3615.71	66515.7	443.86	2110
COEF VAR	0.90	1.12	0.21	0.40
SKEWNESS	6.39	7.32	0.84	0.76
KURTOSIS	57.54	72.76	2.05	1.48
RANGE	2264.4	3220.29	158.52	337
MINIMUM	74.59	75.08	48.32	17.2
.10 QUANTILE	108.29	108.79	76.88	57.1
.25	126.07	127.96	86.19	82.5
.50	161.63	166.20	99.25	107
.75	220.09	228.03	112.71	140
.90	331.77	364.51	124.77	172
MAXIMUM	2339.01	3295.38	206.84	354

CASE IV. TRANSLATED RAYLEIGH DISTRIBUTION

$$Y = -\ 14282.6809 \ln U + 100$$

MOE	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
MEAN	99.79	99.79	95.85	108
STD DEV	11.56	11.58	12.64	38.3
MSE	133.74	134.09	176.96	1524
GED MEAN	99.12	99.11	94.78	100
VARIANCE	133.70	134.05	159.74	1460
COEF VAR	0.12	0.12	0.13	0.35
SKEWNESS	0.15	0.15	- 0.56	0.36
KURTOSIS	0.10	0.08	4.59	0.34
RANGE	70.75	70.25	135.40	248
MINIMUM	67.44	67.46	7.73	20.4
.10 QUANTILE	84.81	84.78	81.18	59.0
.25	92.04	92.00	87.60	82.7
.50	99.48	99.48	96.18	105
.75	107.28	107.12	103.97	132
.90	114.56	114.59	110.99	156
MAXIMUM	138.19	137.71	143.13	268

CASE V. WEIBULL DISTRIBUTION w = 2

$$Y = (-\ 144260504.1 \ln U)^{\frac{1}{2}}$$

DE	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
AN	86.18	85.80	84.25	100
D DEV	5.23	5.19	11.47	19.6
E	218.34	228.61	379.57	382.00
D MEAN	86.03	85.64		98.1
ARIANCE	27.35	26.97	131.51	382
DEF VAR	0.06	0.06	0.14	0.20
KEWNESS	0.029	0.028	- 5.47	- 0.24
URTOSIS	0.28	0.27	38.40	0.17
NGE	32.67	32.46	112.79	121
NIMUM	71.49	71.14	0.0	420
10 QUANTILE	79.37	79.04	78.33	74.2
25	82.87	82.52	81.89	88.5
50	86.29	85.89	85.42	100.0
75	89.50	90.12	88.69	113
90	92.50	92.07	91.56	124
XIMUM	104.16	103.61	112.79	163

CASE VI. WEIBULL DISTRIBUTION $w = 1.5$

$$Y = (-\ 1442695.04 \ln U)^{\frac{2}{3}}$$

MOE	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
MEAN	89.58	89.28	87.51	102
STD DEV	7.07	7.04	12.85	25.7
MSE	158.65	164.46	321.29	664
GED MEAN	89.30	89.00		98.4
VARIANCE	50.07	49.54	165.09	660
COEF VAR	0.08	0.08	0.148	- 0.25
SKEWNESS	0.07	0.07	- 4.27	- 0.04
KURTOSIS	0.22	0.21	27.91	0.11
RANGE	44.55	44.26	125.57	161
MINIMUM	69.36	69.10	0.0	31.5
.10 QUANTILE	80.27	79.99	78.95	67.6
.25	85.39	85.12	83.80	86.4
.50	89.59	89.25	88.60	101
.75	94.37	94.07	93.44	120
.90	98.31	98.00	97.48	133
MAXIMUM	113.91	113.37	125.57	193

CASE VII. WEIBULL DISTRIBUTION $w = .5$

$$Y = (-\ 144.27 \ln U)^2$$

MOE	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
MEAN	166.57	170.05	87.45	146
STD DEV	42.61	44.99	31.55	99.2
MSE	6246.94	6930.96	1153.15	11956
GED MEAN	161.26	164.32	81.72	113
VARIANCE	1815.38	2023.96	995.65	9840
COEF VAR	0.26	0.26	0.36	0.68
SKEWNESS	0.62	0.70	0.76	1.39
KURTOSIS	0.53	0.76	1.48	3.26
RANGE	250.79	258.53	218.72	746
MINIMUM	75.81	78.48	18.67	3.24
.10 QUANTILE	116.20	117.74	49.30	41.2
.25	135.54	137.27	66.56	75.8
.50	163.59	165.51	84.41	126
.75	191.58	196.45	105.80	194
.90	220.85	227.60	127.69	282
MAXIMUM	326.60	337.01	237.40	750

CASE VIII. WEIBULL DISTRIBUTION $w = .1$

$$Y = (-\ 3.623886 \ln U)^{10}$$

MOE	$\hat{C}EP$ 1	$\hat{C}EP$ 2	$\hat{C}EP$ 3	$\hat{C}EP$ 4
MEAN	$2.6\ 10^5$	$3.6\ 10^5$	131.06	$5.1\ 10^4$
STD DEV	$7.8\ 10^5$	$1.1\ 10^6$	335.02	$2.8\ 10^5$
MSE	$6.7\ 10^{11}$	$1.4\ 10^{12}$	$1.13\ 10^5$	$8.2\ 10^{10}$
GED MEAN	$6.0\ 10^4$	$7.4\ 10^4$	85.89	674
VARIANCE	$6.0\ 10^{11}$	$1.3\ 10^{12}$	$1.1\ 10^5$	$7.9\ 10^{10}$
COEF VAR	2.97	3.14	2.56	5.43
SKEWNESS	7.95	8.38	14.16	14.4
KURTOSIS	81.28	89.17	232.51	255
RANGE	9.8 10	$1.4\ 10^7$	6214.98	$5.4\ 10^6$
MINIMUM	461.12	476.59	5.65	$7.2\ 10^6$
.10 QUANTILE	6859.66	7661.19	30.28	4.41
.25	$1.8\ 10^4$	$2.2\ 10^4$	54.43	77.0
.50	$5.9\ 10^4$	$7.4\ 10^4$	93.37	1070
.75	$1.7\ 10^5$	$2.2\ 10^5$	140.28	9280
.90	$5.6\ 10^5$	$7.3\ 10^5$	195.25	$8.3\ 10^4$
MAXIMUM	$9.8\ 10^6$	$1.4\ 10^7$	6220.63	$5.4\ 10^6$

COMPARISON OF ESTIMATORS

USING MEAN SQUARED ERROR AS A CRITERION:

$$MSE[\hat{CEP}] = \hat{\sigma}_{\hat{CEP}} + [E(\hat{CEP}) - CEP]^2$$

CASE NO.	DISTRIBUTION	$\hat{CEP}$ 1	$\hat{CEP}$ 2	$\hat{CEP}$ 3	$\hat{CEP}$ 4
I	RAYLEIGH	136	136	165	1549
II	UNIFORM	109	107	368	1744
III	POWER	2460	83,000	444	2279
IV	TRANSLATED RAY.	134	134	177	1524
V	WEIBULL w = 2.	218	228	379	382
VI	WEIBULL w = 1.5	158	164	321	664
VII	WEIBULL w = .5	6246	6930	1153	11956
VIII	WEIBULL w = .1	$6.77\ 10^{11}$	$6.78\ 10^{11}$	$1.13\ 10^{5}$	$8.17\ 10^{10}$

VII. CONCLUSION

In general, we found the maximum likelihood power estimator $\hat{CEP}$ 3 is better than moment method power estimator, in that it does not have the limitations and exists for all parent distributions we considered. Using minimum mean squared error as a criterion, the estimators based on the power model have inferior performance in the "shallow tailed" cases, but were best in the heavy tailed cases, this is especially true for the w = .1 Weibull case, in which it has less bias than its competitors and has relatively small variance.

We found the Rayleigh based CEP estimators ($\hat{CEP}$ 1) to be possibly the best overall, especially for the shallow tailed cases. The maximum likelihood jackknife Rayleigh estimator ($\hat{CEP}$ 2) had no better performance than the unbiased Rayleigh estimator.

The value of the sample median estimator ($\hat{CEP}$ 4) can be found for small samples with such rapidity that in certain cases the time saved may compensate for the accuracy lost.

In practice, when analysts find several impact points comparatively far away from target, they should choose maximum likelihood power estimator ($\hat{CEP}$ 3). Otherwise they could choose Rayleigh based CEP estimator ($\hat{CEP}$ 1).

APPENDIX 1: MAXIMUM LIKELIHOOD POWER ESTIMATOR

```
      DIMENSION Y(200),S(2001),A(2001)
      DIMENSION CEP(500)
      CALL OVFLOW
      ISEED=470723
      READ (5,100) A0,S0
      READ (5,200) NS,ISTOP,S(1),A(1)
      SN=FLOAT(NS)
      FA=-1.0/A0
      KS=500
      DO 1 KI=1, KS
C
C     GENERATE POWER SQUARE DEVIATE
      DO 5 N=1,NS
      Y(N)=0.0
      CALL RANDOM (ISEED,U,1)
      Y(N)=S0*(U**FA-1.0)
    5 CONTINUE
C
C     SOLVE NONLINEAR EQUATION UNDER NEWTON'S SECOND METHOD
      DO 10 I=1,ISTOP
      ITERS=I
      A(I+1)=0.0
      S(I+1)=0.0
      CALL BF (S(I),Y,NS,B)
      CALL CF (S(I),Y,NS,C)
      CALL DF (S(I),Y,NS,D)
      E=ALOG(S(I))
      UX=SN/A(I)+SN*E-B
      VX=SN*A(I)/S(I)-(A(I)+1.0)*D
      WX=-SN/(A(I)**2)
      XX=SN/S(I)-D
      ZX=-SN*A(I)/(S(I)**2)+(A(I)+1.0)*C
      DET=WX*ZX-XX**2
C
C     CHECK THE DETERMINATION
      IF (DET.EQ.0.0) GO TO 99
      UU=ZX*UX-XX*VX
      VV=WX*VX-XX*UX
C
C     WHETHER TO STOP THE ITERATION
      CHI=-(UX*UU+VX*VV)/DET
      IF (CHI.LE.0.040) GO TO 88
      A(I+1)=A(I)-UU/DET
      S(I+1)=S(I)-VV/DET
      IF (S(I+1).GT.0.0) GO TO 10
      S(I+1)=S(I)/2.0
   10 CONTINUE
   88 AOUT=A(ITERS)
      SOUT=S(ITERS)
C
C    ESTIMATE CEP
      AK=2**(1.0/AOUT)-1.0
      CEP(KI)=(SOUT*AK)**0.5
      WRITE (6,400) SOUT,AOUT,ITERS,CEP(KI)
    1 CONTINUE
      CALL HISTG (CEP,KS,0)
   99 STOP
      END


      SUBROUTINE BF (S,Y,NS,B)
      DIMENSION Y(200)
      B=0.0
      DO 30 N=1,NS
      B=B+ALOG(S+Y(N))
   30 CONTINUE
```

```
      RETURN
      END

      SUBROUTINE CF (S,Y,NS,C)
      DIMENSION Y(200)
      C=0.0
      DO 40 N=1,NS
      C=C+1.0/(S+Y(N))**2
   40 CONTINUE
      RETURN
      END

      SUBROUTINE DF (S,Y,NS,D)
      DIMENSION Y(200)
      D=0.0
      DO 50 N=1,NS
      D=D+1.0/(S+Y(N))
   50 CONTINUE
      RETURN
      END
```

APPENDIX 2: RAYLEIGH UNBIASED AND JACKKNIFE ESTIMATOR

```
      DIMENSION Y(50,50),SP(50),PSE(50),CEP(500)
      DIMENSION RAY(500)
C
C     GENERATE RAYLEIGH DEVIATE
      CALL OVFLOW
      ISEED=470723
C
C     READ IN ORIGINAL POP. PARAMETER VO , TOTAL SAMPLE SIZE
C      NS, SUBGROUP SAMPLE SIZE KS
      READ (5,100)  VO,NS,KS
      CPO=1.17741002*SQRT(VO)
      MS=NS/KS
      SM=FLOAT(MS)
      SN=FLOAT(NS)
      SK=FLOAT(KS)
      WRITE (6,200) VO,CPO,NS,MS,KS
      KE=500
      DO 1 KRE=1,KE
      DO 5 M=1,MS
      DO 5 K=1,KS
      Y(M,K)=0.0
      CALL RANDOM (ISEED,U,1)
      Y(M,K)=10000.0*(1.0/U-1.0)
    5 CONTINUE
C
C     ESTIMATE THE PARAMETER BASED ON ALL DATA
      T=0.0
      DO 10 M=1,MS
      DO 10 K=1,KS
      T=T+Y(M,K)
   10 CONTINUE
      TM=SQRT(T/(SN*2.0))
C
C     MAXIMUM LIKELIHOOD UNBIASED ESTIMATER
      RAY(KRE)=1.1848*TM
C
C     ESTIMATE THE PARAMETER BASED ON ALL DATA EXCEPT THOSE
C     IN THE ITH GROUP
      DO 20 M=1,MS
      S=0.0
      DO 25 I=1,MS
      IF (I.EQ.M) GO TO 25
      DO 26 K=1,KS
      S=S+Y(I,K)
   26 CONTINUE
   25 CONTINUE
      SP(M)=SQRT(S/((SN-SK)*2.0))
   20 CONTINUE
C
C     COMPUTE THE MS PSEUDO VALUES
      DO 40 M=1,MS
      PSE(M)=SM*TM-(SM-1.0)*SP(M)
   40 CONTINUE
C
C     ESTIMATE JACKKNIFE PARAMETER
      ACK=0.0
      DO 45 M=1,MS
      ACK=ACK+PSE(M)
   45 CONTINUE
      VJACK=ACK/SM
      SCK=0.0
C
C     COMPUTER THE CONFIDENCE INTERVAL
      DO 50 I=1,MS
      SCK=SCK+(PSE(I)-VJACK)**2
   50 CONTINUE
```

```
      SVJA=SCK/(SM*(SM-1.0))
      UPB=VJACK-2.0*SQRT(SVJA)
      PLB=VJACK+2.0*SQRT(SVJA)
C
C     ESTIMATE CEP FROM JACKKNIFE PARAMETER
      CEP(KRE)=1.1774*VJACK
      WRITE (6,300) VJACK,UPB,PLB,CEP(KRE)
    1 CONTINUE
      CALL HISTG (RAY,KE,0)
      CALL HISTG (CEP,KE,0)
      STOP
      END
```

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